Fractional Permutations

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Introduction. Markov matrices are real square matrices

$$\mathbb{M} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \quad : \quad p_{ij} = \text{probability } i \leftarrow j$$

with elements that (as conditional probabilities) fall within the interval $[0, 1]$ and that possess the property that if the column vector $\mathbf{p}$ is stochastic (real elements $\in [0, 1]$ that sum to unity) then so is $\mathbb{M}\mathbf{p}$, which entails that the columns of $\mathbb{M}$ sum to unity. Discrete Markov processes proceed

$$\mathbf{p} \to \mathbb{M}\mathbf{p} \to \mathbb{M}^2\mathbf{p} \to \mathbb{M}^3\mathbf{p} \to \cdots$$

We have the generalized spectral resolution of $\mathbb{M}$ (biorthogonal generalization of the standard spectral resolution that is available when $\mathbb{M}$ is symmetric; *i.e.*, when the process is subject to "detailed balance")

$$\mathbb{M} = \lambda_1\mathbb{P}_1 + \lambda_2\mathbb{P}_2 + \cdots + \lambda_\nu\mathbb{P}_\nu \quad : \quad \nu \leqslant n \tag{1}$$

where $\{\lambda_1, \lambda_1, \ldots, \lambda_\nu\}$ are the distinct eigenvalues of $\mathbb{M}$, and $\{\mathbb{P}_1, \mathbb{P}_1, \ldots, \mathbb{P}_\nu\}$ are a complete set of orthogonal projection matrices that project onto the respective eigenspaces of $\mathbb{M}$. If none of the eigenvalues vanish (all Markov matrices are in fact non-singular) we can introduce

$$\mathbb{L} = \log\mathbb{M} = \log\lambda_1\cdot\mathbb{P}_1 + \log\lambda_2\cdot\mathbb{P}_2 + \cdots + \log\lambda_\nu\cdot\mathbb{P}_\nu \tag{2}$$

and write

$$\mathbb{M} = \exp\mathbb{L} \tag{3}$$

But eigenvalues of $\mathbb{M}$ will occasionally be complex (occurring in conjugate pairs), and so therefore will be the associated eigenvectors and projection matrices. And even when $\mathbb{M}$ is symmetric, some of the eigenvalues—all of them now real—may be negative. In all such cases $\log\lambda$ is complex, and it is only by delicate conspiracy that the constructions (3) manage to be real.

By natural generalization of (3) we have

$$\mathbb{M}(t) = \exp\{t\mathbb{L}\} \quad : \quad t \text{ to be thought of as "time"} \tag{4}$$

which gives back the real matrices $\mathbb{M}^t$ at positive integral values of t, and at non-integral times provides a *continuous interpolation between* those integral powers of $\mathbb{M}$. But at non-integral times $\mathbb{M}(t)$ will in general (*i.e.*, except when all eigenvalues of $\mathbb{M}$ are positive real numbers) be complex, with the consequence that the vectors $\mathbf{p}(t) = \mathbb{M}(t)\,\mathbf{p}$ are not interpretable as stochastic vectors.

We note in passing that the set of all Markov matrices (of given dimension) is multiplicatively closed (yet not a group, because $\mathbb{M}^{-1}$ is generally not Markovian: it has elements of the wrong sign, and/or that fall outside of $[0, 1]$). From

$$\mathbb{M}(t+\tau) = (\mathbb{I} + \tau\mathbb{L})\cdot\mathbb{M}(t)$$

we see that $\mathbb{M}(t)$ and $\mathbb{M}(t+\tau)$ (here $\tau = \delta t$ is infinitesimal) will both be Markovian if and only if $\mathbb{L}$ is a "Kirchhoff" matrix (real, with columns that sum to zero), but must in general expect $\mathbb{L}$ to be a "complex Kirchhoff " matrix.

Permutation matrices $\mathbb{P}$ are Markov matrices endowed with the special property that all elements are either 1 or 0, which requires that a solitary 1 appears in every row and column. Let $\{\mathcal{P}\}$ refer to the set of all permutations of n symbols/objects. The set $\{\mathcal{P}\}$ has $n!$ elements, of which one is the identity permutation $\mathcal{I}$, and is closed under composition, from which it follows that for every individual $\mathcal{P}$ there is a μ such that $\mathcal{P}^\mu = \mathcal{I}$: $\{\mathcal{P}\}$ is a group, every element of which is cyclic, with $\mathcal{P}^{-1} = \mathcal{P}^{\mu-1}$. Properties of the group generated by any individual $\mathcal{P}$ follow from those of its cyclic subgroups; in particular, the period of $\mathcal{P}$ is the least common multiple of the periods of its subgroups. All of which is reflected in properties of the set $\{\mathbb{P}\}$ of $n\times n$ matrix representations of $\{\mathcal{P}\}$, which follow from properties of the cyclic permutation matrices

$$\mathbb{C}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbb{C}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbb{C}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \ldots$$

that one can use to construct matrices such as

$$\begin{pmatrix} \mathbb{C}_2 & \mathbb{O} \\ \mathbb{O} & \mathbb{C}_3 \end{pmatrix}, \quad \mathbb{C}_2 \otimes \mathbb{C}_3 = \begin{pmatrix} \mathbb{O} & \mathbb{C}_3 \\ \mathbb{C}_3 & \mathbb{O} \end{pmatrix}, \quad \textit{etc.}$$

My objective will be to construct matrices $\mathbb{C}_n(t)$ that interpolate between the integral powers of such matrices. I begin by reviewing the essentials of the tool that supports the discussion.

Generalized spectral decomposition. Let $\mathbb{X}$ be an arbitrary (real or complex, singular or non-singular) $n \times n$ matrix, let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be its (possibly degenerate) spectrum, and let $\{|a_1), |a_2), \ldots, |a_n)\}$ be an eigenbasis in the (generally complex) vector space $\mathcal{V}_n$ (with normalization not assumed). From $\{|a_j)\}$ and their transposed conjugates $\{(a_i|\}$ construct the hermitian matrix

$$\|h_{ij}\| \quad : \quad h_{ij} = (a_i|a_j)$$

(which is non-singular by linear independence of the $\{|a_j)\}$) and its inverse

$$\|h^{ij}\| \quad : \quad h^{ik}h_{kj} = \delta^i{}_j$$

The generic element $|x)\in\mathcal{V}_n$ can be developed

$$|x) = |a_j)\,x^j$$

which gives

$$(a_k|x) = h_{kj}x^j$$

Therefore

$$h^{ik}(a_k|x) = h^{ik}h_{kj}x^j = \delta^i{}_j x^j = x^i$$

giving

$$|x) = |a_i)h^{ik}(a_k|x) \quad : \quad \text{all } |x)$$

from which we conclude that

$$|a_i)\,g^{ij}(a_j| = \mathbb{I}$$

Introduce now into $\mathcal{V}_n$ a second (generally non-orthogonal) "reciprocal basis" with elements

$$|A^j) = |a_i)h^{ij} \quad \text{similarly} \quad (A^i| = h^{ij}(a_j|$$

which supply these alternative constructions of the unit matrix:

$$|A^i)(a_i| = |A^i)h_{ij}(A^j| = |a_j)(A^j| = \mathbb{I}$$

Moreover

$$(A^i|a_j) = g^{ik}(a_k|a_j) = g^{ik}g_{kj} = \delta^i{}_j$$

which is to say:

$$|A^i) \perp \text{all } |a_j) \quad : \quad j \neq i$$

The reciprocal bases $\{|a_i)\}$ and $\{|A^j)\}$ are said to be "biorthogonal."

Look now to the matrices

$$\mathbb{P}_i = |a_i)(A^i| \quad : \quad \text{no summation on } i$$

where the index placement on $\mathbb{P}_i$ is merely conventional (intended to convey no transformation-theoretic meaning) and the notation $\mathbb{P}$ is intended here to suggest not "permutation" but "projection." It follows from preceding remarks that the $\{\mathbb{P}_i\}$ comprise a complete set of orthogonal projection operators:

$$\textstyle\sum_i \mathbb{P}_i = \mathbb{I}$$

$$\mathbb{P}_i\mathbb{P}_j = |a_i)(A^i|a_j)(A^j| = |a_i)\delta^i{}_j(A^j| = \begin{cases} \mathbb{P}_i & : \quad i = j \\ \mathbb{O} & : \quad i \neq j \end{cases}$$

They project onto 1-spaces (rays); specifically

$$\left.\begin{array}{rl} \text{right action:} & \mathbb{P}_i|x) = |a_i)\,x^i \\ \text{left action:} & (x|\mathbb{P}_i = x_i\,(A^i| \end{array}\right\} \quad : \quad \text{no summation on } i$$

We now have

$$\begin{aligned} \mathbb{X} &= \mathbb{X}\mathbb{I} \\ &= \textstyle\sum_i \mathbb{X}\mathbb{P}_i \\ &= \textstyle\sum_i \mathbb{X}|a_i)(A^i| = \sum_i \lambda_i|a_i)(A^i| = \sum_i \lambda_i\mathbb{P}_i \end{aligned} \tag{5}$$

which is the "generalized spectral representation" of $\mathbb{X}$, and gives back the standard spectral representation in cases where the eigenvectors of $\mathbb{X}$ are orthogonal (and can be assumed to have been normalized).

More generally, let $\mathbb{W}$ be an *arbitrary* $n \times n$ square matrix. We are in position now to write

$$\begin{aligned}\mathbb{W} &= \mathbb{I}\,\mathbb{W}\,\mathbb{I}\\ &= \textstyle\sum_{ij} \mathbb{P}_i\,\mathbb{W}\,\mathbb{P}_j\\ &= \textstyle\sum_{ij} |a_i)(A^i|\mathbb{W}|a_j)(A^j|\\ &= \textstyle\sum_{ij} w^i{}_j |a_i)(A^j| \quad \text{where} \quad w^i{}_j = (A^i|\mathbb{W}|a_j)\end{aligned} \tag{6}$$

Here $\mathbb{W}$ is displayed as a weighted linear combination of the n^2-population of matrices

$$\mathbb{F}_i{}^j = |a_i)(A^j| \quad : \quad \text{gives back } \mathbb{P}_i \text{ when } i = j$$

(these provide a "basis in the space of matrices") and $\|m^i{}_j\|$ provides, with respect to the non-orthogonal $\{|a_i)\}$-basis, the matrix representation of $\mathbb{W}$; it permits $|x) \rightarrow |\tilde{x}) = \mathbb{W}|x)$ to be represented

$$x^i \rightarrow \tilde{x}^i = m^i{}_j x^j$$

We note in passing that the $\mathbb{F}$-matrices are tracewise orthogonal

$$\mathrm{tr}(\mathbb{F}_i{}^j\,\mathbb{F}_k{}^l) = \delta_{ik}{}^{jl}$$

so (6) can be written as a "generalized Fourier identity"

$$\mathbb{W} = \textstyle\sum_{ij} w^i{}_j \mathbb{F}_i{}^j \quad \text{with} \quad w^i{}_j = \mathrm{tr}(\mathbb{F}_i{}^j\,\mathbb{W})$$

Fractional permutation in the simplest case. We look to the case

$$\mathbb{C}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\mathbb{C}_2$ is a symmetric real matrix, so standard spectral decomposition suffices, but to achieve consistency with subsequent discussion of higher-order cases we adopt the generalized protocol, and adopt also the ordering/normalization conventions of *Mathematica*.

The eigenvalues of $\mathbb{C}_2$ are square roots of unity: $\{\lambda_1, \lambda_2\} = \{-1, 1\}$. The respective eigenvectors are

$$|a_1) = \begin{pmatrix} -1 \\ +1 \end{pmatrix}, \quad |a_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore

$$\|h_{ij}\| = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \|h^{ij}\| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$|A^1) = \begin{pmatrix} -1/2 \\ +1/2 \end{pmatrix}, \quad |A^2) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$\mathbb{P}_1 = \begin{pmatrix} +1/2 & -1/2 \\ -1/2 & +1/2 \end{pmatrix}, \quad \mathbb{P}_2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

(the complete, orthogonal and projective properties of which check out) and we obtain finally the manifestly correct statement

$$\mathbb{C}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (-1)\,\mathbb{P}_1 + (+1)\,\mathbb{P}_1$$

But $\log(-1) = i\pi$ and $\log(+1) = 0$ so we have

$$\mathbb{L}_2 \equiv \log \mathbb{C}_2 = i\pi\mathbb{P}_1 = \tfrac{i\pi}{2}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \tag{7}$$

giving finally (with assistance here of the command `MatrixExp[]`)

$$\mathbb{C}_2(t) = \exp\{t\mathbb{L}_2\} = \tfrac{1}{2}\begin{pmatrix} 1+e^{i\pi t} & 1-e^{i\pi t} \\ 1-e^{i\pi t} & 1+e^{i\pi t} \end{pmatrix} \tag{8}$$

We note/verify that (i) the imaginary matrix $\mathbb{L}_2$ possesses the Kirtchhoff property (columns add to 0); (ii) $\mathbb{C}_2(t)$ is complex except when t is an integer; (iii) $\mathbb{C}_2(t)$ composes by the simple rule

$$\mathbb{C}_2(u)\boldsymbol{\cdot}\mathbb{C}_2(v) = \mathbb{C}_2(u+v)$$

from which by $\mathbb{C}_2(2) = \mathbb{I}$ it follows that (iv) $\mathbb{C}_2(t)$ is periodic (cyclic):

$$\mathbb{C}_2(t+2) = \mathbb{C}_2(t)$$

(v) $\mathbb{C}_2(t)$ is non-singular for all t:

$$\det \mathbb{C}_2(t) = \exp\{\mathrm{tr}(t\mathbb{L}_2)\} = e^{i\pi t}$$

which is real if and only if $t = p$ (an integer), and when real is proper/improper (± 1, in permutation language is even/odd) according as p is even or odd; (vi) the eigenvalues $\{1, e^{i\pi t}\}$ of $\mathbb{C}_2(t)$ are the exponentiated eigenvalues of and $t\mathbb{L}_2 = \log \mathbb{C}_2(t)$; (vii) $\mathbb{C}_2(t)\mathsf{p}$ is complex (therefore not interpretable as a stochastic vector) unless t is an integer, but in all cases its elements sum to unity; $(viii)$ $\mathbb{C}_2(t)$ is unitary:

$$\mathbb{C}_2^{\text{adjoint}}(t) = \mathbb{C}_2^{\text{inverse}}(t) = \mathbb{C}_2(-t)$$

Fractional permutation in the next simplest case. The matrix

$$\mathbb{C}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is *not* symmetric: the full force of the generalized spectral resolution comes now indispensably into play. The eigenvalues of $\mathbb{C}_3$ are the cube roots of unity

$$\begin{aligned}\{\lambda_1,\lambda_2,\lambda_3\} &= \{(-1)^{2/3}, -(-1)^{1/3}, 1\}\\ &= \{e^{i\,\omega}, e^{i\,2\omega} = e^{-i\,\omega}, e^{i\,3\omega} = 1\} \;:\; \omega = 2\pi/3\\ &= \big\{\tfrac{1}{2}(-1+i\sqrt{3}), \tfrac{1}{2}(-1-i\sqrt{3}), 1\big\}\end{aligned}$$

and the associated eigenvectors can be described

$$|a_1) = \begin{pmatrix} -e^{i\,\omega}-1\\ e^{i\,\omega}\\ 1\end{pmatrix},\quad |a_2) = \begin{pmatrix} -e^{-i\,\omega}-1\\ e^{-i\,\omega}\\ 1\end{pmatrix},\quad |a_3) = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

(We note, for what it's worth, that the elements of $|a_1)$ and $|a_2)$ sum to zero.) So $\|h_{ij}\| = 3\mathbb{I}$, $\|h^{ij}\| = \frac{1}{3}\mathbb{I}$ and (after major simplification of the results announced by *Mathematica*)

$$\mathbb{P}_1 = \tfrac{1}{3}\begin{pmatrix} 1 & e^{i\,\omega} & e^{-i\,\omega}\\ e^{-i\,\omega} & 1 & e^{i\,\omega}\\ e^{i\,\omega} & e^{-i\,\omega} & 1\end{pmatrix},\quad \mathbb{P}_2 = \tfrac{1}{3}\begin{pmatrix} 1 & e^{-i\,\omega} & e^{+i\,\omega}\\ e^{i\,\omega} & 1 & e^{-i\,\omega}\\ e^{-i\,\omega} & e^{i\,\omega} & 1\end{pmatrix}$$

$$\mathbb{P}_3 = \tfrac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

We confirm that

$$\mathbb{C}_3 = \lambda_1\mathbb{P}_1 + \lambda_2\mathbb{P}_2 + \lambda_3\mathbb{P}_3$$

The log spectrum is $\{i\omega, -i\omega, 0\}$ so

$$\mathbb{L}_3 \equiv \log\mathbb{C}_3 = i\omega(\mathbb{P}_1 - \mathbb{P}_2) = \omega\cdot\tfrac{1}{\sqrt{3}}\begin{pmatrix}0&-1&1\\1&0&-1\\-1&1&0\end{pmatrix} \quad:\quad \omega = \tfrac{1}{3}\cdot 2\pi$$

This (surprisingly?) is a *real* antisymmetric matrix, the generator of a ω-rotation about the $(1,1,1)$ axis; the (trivially unitary) rotation matrix

$$\mathbb{C}_3(t) = \exp\{t\mathbb{L}_3\}$$

is familiar as the rotation that cyclically permutes orthogonal axes in 3-space. It sends

$$\begin{aligned}\begin{pmatrix}a\\b\\c\end{pmatrix} &\to \tfrac{1}{3}\begin{pmatrix} a(1+2\cos\theta) + b(1-\cos\theta-\sqrt{3}\sin\theta) + c(1-\cos\theta+\sqrt{3}\sin\theta)\\ b(1+2\cos\theta) + c(1-\cos\theta-\sqrt{3}\sin\theta) + a(1-\cos\theta+\sqrt{3}\sin\theta)\\ c(1+2\cos\theta) + a(1-\cos\theta-\sqrt{3}\sin\theta) + b(1-\cos\theta+\sqrt{3}\sin\theta)\end{pmatrix}\\ &= \begin{pmatrix}c\\a\\b\end{pmatrix}\quad \text{at } \theta\equiv\omega t = \tfrac{1}{3}2\pi\\ &= \begin{pmatrix}b\\c\\a\end{pmatrix}\quad \text{at } \theta\equiv\omega t = \tfrac{2}{3}2\pi\end{aligned}$$

and provides real-valued (!) interpolations between those discrete permutations. $\mathbb{C}_3(t)$ possess properties analogous to—but differing in certain particulars from —those previously listed for $\mathbb{C}_2(t)$: (*i*) the matrix $\mathbb{L}_2$ possesses the Kirchhoff property; (*ii*) $\mathbb{C}_3(t)$ is now real for all values of t; (*iii*) $\mathbb{C}_3(t)$ composes by the simple rule $\mathbb{C}_3(u)\cdot\mathbb{C}_3(v) = \mathbb{C}_3(u+v)$, from which by $\mathbb{C}_3(3) = \mathbb{I}$ it follows that (*iv*) $\mathbb{C}_3(t)$ is periodic (cyclic): $\mathbb{C}_3(t+3) = \mathbb{C}_3(t)$; (*v*) $\mathbb{C}_3(t)$ is non-singular

$$\det \mathbb{C}_3(t) = \exp\{\mathrm{tr}(t\mathbb{L}_3)\} = e^0 = 1$$

—and, indeed, proper—for all t: all cyclic permutations—even fractional permutations—of three objects are even; (*vi*) the eigenvalues $\{e^{i\omega t}, e^{-i\omega t}, 1\}$ of $\mathbb{C}_3(t)$ are the exponentiated eigenvalues of $t\mathbb{L}_3 = \log \mathbb{C}_3(t)$; (*vii*) $\mathbb{C}_3(t)\,\mathbf{p}$ is real, and is invariably interpretable as a stochastic vector; (*viii*) $\mathbb{C}_3(t)$ is a real rotation matrix: $\mathbb{C}_3^{\text{transpose}}(t) = \mathbb{C}_3^{\text{inverse}}(t) = \mathbb{C}_3(-t)$, so trivially unitary.

I digress to make more explicit the rotational action of $\mathbb{C}_3(t)$.[1] Exploiting the orthogonal projectivity of the $\mathbb{P}$-matrices, we have

$$\begin{aligned}
\mathbb{C}_3(t) &= \exp\big\{t\,(i\omega\,\mathbb{P}_1 - i\omega\,\mathbb{P}_2 + 0\,\mathbb{P}_3)\big\}\\
&= e^{i\,\omega\,t}\,\mathbb{P}_1 + e^{-i\,\omega\,t}\,\mathbb{P}_2 + \mathbb{P}_3\\
&= \cos(\omega t)\cdot(\mathbb{P}_1 + \mathbb{P}_2) + i\,\sin(\omega t)\cdot(\mathbb{P}_1 - \mathbb{P}_2) + \mathbb{P}_3
\end{aligned}$$

The complex matrices $\{\mathbb{P}_1, \mathbb{P}_2\}$ were seen to be complex conjugates of each other, so

$$\begin{aligned}
\mathbb{Q} = \ (\mathbb{P}_1 + \mathbb{P}_2) &= \ \tfrac{1}{3}\begin{pmatrix} +2 & -1 & -1\\ -1 & +2 & -1\\ -1 & -1 & +2\end{pmatrix} = \mathbb{I} - \mathbb{P}_3\\
\mathbb{A} = i\,(\mathbb{P}_1 - \mathbb{P}_2) &= \tfrac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & +1\\ +1 & 0 & -1\\ -1 & +1 & 0\end{pmatrix}
\end{aligned}$$

are both real. We have only to note that $\mathbb{A} = \mathbb{A}\mathbb{Q}$ to obtain

$$\mathbb{C}_3(t) = \mathbb{P}_3 + \big\{\cos(\omega t)\,\mathbb{I} + \sin(\omega t)\,\mathbb{A}\big\}\cdot\mathbb{Q}$$

When applied to a 3-vector $\boldsymbol{r}$ the leading $\mathbb{P}_3$ term projects out the t-independent component $\boldsymbol{r}_\parallel$ of $\boldsymbol{r} = \boldsymbol{r}_\parallel + \boldsymbol{r}_\perp$ that is parallel to the spin axis $(1,1,1)$, while the trailing term causes the normal component $\boldsymbol{r}_\perp$ to rotate with angular velocity ω.

Fractional cyclic permutations of 4th order. Marked differences emerged when we advanced from cyclic permutations of second order to permutations of third order. It is natural to speculate that this may have partly to do with the fact

[1] Here I borrow from the discussion of n-dimensional rotation matrices that appears on pages 14–18 of "Extrapolated interpolation theory" (April 1997), which was adapted from and "What does an N-dimensional rotation look like?" (notes for a Reed College Math Seminar presented on 14 February 1980; Transformational Physics & Physical Geometry 1971–1983).

that iterated cyclic permutations of even order proceed odd-even-odd-even. . . , while those of even order proceed even-even-even-even. . . I look, therefore, to some higher-order cases. The relevant *Mathematica* commands are essentially order independent, but the results become rapidly too complex to transcribe. I must be content, therefore, to leave some of them in the notebook where they were born, and to report only their most salient features.

We look to the case

$$\mathbb{C}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The eigenvalues of $\mathbb{C}_4$ are the fourth roots of unity (which *Mathematica* presents in an oddly unnatural order)

$$\begin{aligned}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} &= \{-1, i, -i, 1\} \\ &= \big\{e^{i\pi}, e^{i\pi/2}, e^{-i\pi/2}, e^0\big\}\end{aligned}$$

The associated eigenvectors are

$$|a_1) = \begin{pmatrix} -1 \\ +1 \\ -1 \\ +1 \end{pmatrix}, \quad |a_2) = \begin{pmatrix} -i \\ -1 \\ +i \\ +1 \end{pmatrix}, \quad |a_3) = \begin{pmatrix} +i \\ -1 \\ -i \\ +1 \end{pmatrix}, \quad |a_4) = \begin{pmatrix} +1 \\ +1 \\ +1 \\ +1 \end{pmatrix}$$

(Again we note that the elements of the first three sum to zero.) So we have $\|h_{ij}\| = 4\mathbb{I}$ and $\|h^{ij}\| = \frac{1}{4}\mathbb{I}$ and are led to the complete quartet of orthogonal complex projection matrices

$$\mathbb{P}_1 = \tfrac{1}{4}\begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{pmatrix}, \quad \mathbb{P}_2 = \tfrac{1}{4}\begin{pmatrix} +1 & +i & -1 & -i \\ -i & +1 & +i & -1 \\ -1 & -i & +1 & +i \\ +i & -1 & -i & +1 \end{pmatrix}$$

$$\mathbb{P}_3 = \tfrac{1}{4}\begin{pmatrix} +1 & -i & -1 & +i \\ +i & +1 & -i & -1 \\ -1 & +i & +1 & -i \\ -i & -1 & +i & +1 \end{pmatrix}, \quad \mathbb{P}_4 = \tfrac{1}{4}\begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & +1 \end{pmatrix}$$

The log spectrum of $\mathbb{C}_4$ is $\{i\pi, \frac{1}{2}i\pi, -\frac{1}{2}i\pi, 0\}$ so

$$\begin{aligned}\mathbb{L}_4 = \log\mathbb{C}_4 &= i\pi\big\{\mathbb{P}_1 + \tfrac{1}{2}\mathbb{P}_2 - \tfrac{1}{2}\mathbb{P}_3\big\} \\ &= \tfrac{\pi}{4}\left\{\begin{pmatrix} 0 & -1 & 0 & +1 \\ +1 & 0 & -1 & 0 \\ 0 & +1 & 0 & -1 \\ -1 & 0 & +1 & 0 \end{pmatrix} + i\begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{pmatrix}\right\}\end{aligned}$$

It is clear by inspection that $\mathbb{L}_4$ is an anti-hermitian (complex) Kirchhoff matrix, so

$$\mathbb{C}_4(t) = \exp\{t\mathbb{L}_4\}$$

is a unitary complex Markov (cyclic permutation) matrix. The composition law $\mathbb{C}_4(u)\cdot\mathbb{C}_4(v) = \mathbb{C}_4(u+v)$ is (as always) obvious, and by $\mathbb{C}_4(4) = \mathbb{I}$ leads to the periodicity statement $\mathbb{C}_4(t+4) = \mathbb{C}_4(u)$. From the general identity

$$\det(e^{\mathbb{A}}) = e^{\operatorname{tr}\mathbb{A}}$$

we have

$$\det \mathbb{C}_4(t) = e^{i\pi t} = \begin{cases} +1 & : \quad t \text{ an even integer} \\ -1 & : \quad t \text{ an odd integer} \end{cases}$$

Exponentiation of

$$t\mathbb{L}_4 = i\omega t\{2\mathbb{P}_1 + \mathbb{P}_2 - \mathbb{P}_1 + 0\mathbb{P}_4\} \quad : \quad \omega = \pi/2$$

gives

$$\begin{aligned} \mathbb{C}_4(t) &= e^{2i\omega t}\mathbb{P}_1 + e^{i\omega t}\mathbb{P}_2 + e^{-i\omega t}\mathbb{P}_3 + e^0\mathbb{P}_4 \\ &= e^{i\omega t}\big\{e^{i\omega t}\mathbb{P}_1 + e^{-i\omega t}\mathbb{P}_4\big\} + \big\{e^{i\omega t}\mathbb{P}_2 + e^{-i\omega t}\mathbb{P}_3\big\} \end{aligned}$$

which can be written

$$\mathbb{C}_4(t) = e^{i\omega t}\big\{\cos(\omega t)\,\mathbb{I} + \sin(\omega t)\mathbb{A}_1\big\}\mathbb{Q}_1 + \big\{\cos(\omega t)\,\mathbb{I} + \sin(\omega t)\mathbb{A}_2\big\}\mathbb{Q}_2$$

where

$$\begin{aligned} \mathbb{Q}_1 &= \quad \mathbb{P}_1 + \mathbb{P}_4 \\ \mathbb{A}_1 &= i\,(\mathbb{P}_1 - \mathbb{P}_4) \\ \mathbb{Q}_2 &= \quad \mathbb{P}_2 + \mathbb{P}_3 \\ \mathbb{A}_2 &= i\,(\mathbb{P}_2 - \mathbb{P}_4) \end{aligned}$$

entail $\mathbb{A}_1\mathbb{Q}_1 = \mathbb{A}_1$, $\mathbb{A}_2\mathbb{Q}_2 = \mathbb{A}_2$. The matrices

$$\mathbb{Q}_1 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad \mathbb{Q}_2 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

comprise a complete set of symmetric orthogonal projection matrices, that by $\operatorname{tr}\mathbb{Q}_1 = \operatorname{tr}\mathbb{Q}_2 = 2$ project onto orthogonal 2-spaces in the complex space $\mathcal{V}_4$. The matrices

$$\mathbb{A}_1 = -i\,\tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbb{A}_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

are imaginary antihermitian and real antisymmetric, respectively—generators by exponentiation of matrices that are respectively unitary/rotational. Their properties are similar in this respect:

$$\mathbb{A}_1\mathbb{A}_1 = -\mathbb{Q}_1, \quad \mathbb{A}_2\mathbb{A}_2 = -\mathbb{Q}_2$$

It follows that the preceding description of $\mathbb{C}_4(t)$ can be formulated

$$\mathbb{C}_4(t) = e^{i\,\omega\,t}\cdot e^{\omega t\mathbb{A}_1}\mathbb{Q}_1 + e^{\omega t\mathbb{A}_2}\mathbb{Q}_2$$

where the first term executes a unitary transformation on the eigenplane of $\mathbb{Q}_1$, and the second a rotation on the orthogonal eigenplane of $\mathbb{Q}_2$. The interpolating matrix $\mathbb{C}_4(t)$ is complex except at integral values of t, when it reproduces the values of $(\mathbb{C}_4)^p$.

Fractional cyclic permutations of 5th order. We look finally to

$$\mathbb{C}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

of which the eigenvalues are fifth roots of unity:

$$\begin{aligned}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\} &= \{(-1)^{4/5}, -(-1)^{1/5}, 1, (-1)^{2/5}, -(-1)^{3/5}\} \\ &= \{e^{i2\,\omega}, e^{i\,3\,\omega}, 1, e^{i\,\omega}, e^{i4\,\omega}\} \quad : \quad \omega = 2\pi/5\end{aligned}$$

The associated eigenvectors[2] are

$$|a_1) = \begin{pmatrix} e^{3i\,\omega} \\ e^{i\,\omega} \\ e^{4i\,\omega} \\ e^{2i\,\omega} \\ 1 \end{pmatrix}, \ |a_2) = \begin{pmatrix} e^{2i\,\omega} \\ e^{4i\,\omega} \\ e^{i\,\omega} \\ e^{3i\,\omega} \\ 1 \end{pmatrix}, \ |a_3) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ |a_4) = \begin{pmatrix} e^{4i\,\omega} \\ e^{3i\,\omega} \\ e^{2i\,\omega} \\ e^{i\,\omega} \\ 1 \end{pmatrix}, \ |a_5) = \begin{pmatrix} e^{i\,\omega} \\ e^{2i\,\omega} \\ e^{3i\,\omega} \\ e^{4i\,\omega} \\ 1 \end{pmatrix}$$

---

[2] To construct the eigenvectors *Mathematica* has to solve quintics, so displays the results as complex numerics. I have used

$$\begin{aligned} e^{i\,\omega} &= 0.309017 + i\,0.951057 \\ e^{2i\,\omega} &= -0.809017 + i\,0.587785 \\ e^{3i\,\omega} &= -0.809017 - i\,0.587785 \\ e^{4i\,\omega} &= 0.309017 - i\,0.951057 \end{aligned}$$

to infer the symbolic meaning of the numerics.

and are, as in all previous cases, orthogonal: $\|h_{ij}\| = 5\mathbb{I}$, $\|h^{ij}\| = \frac{1}{5}\mathbb{I}$. The resulting $\mathbb{P}$-matrices are—except for

$$\mathbb{P}_3 = \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

—complex hermitian and have columns that sum to zero. They all have the form $\frac{1}{5}\big($a matrix in which every eigenvalue appears in every row & column$\big)$, but it would serve no useful purpose to write them down. They comprise a complete orthogonal set of projection matrices.[3] From the generalized spectral resolution

$$\mathbb{C}_5 = e^{2i\,\omega}\mathbb{P}_1 + e^{-2i\,\omega}\mathbb{P}_2 + \mathbb{P}_3 + e^{i\,\omega}\mathbb{P}_4 + e^{-i\,\omega}\mathbb{P}_5$$

we are led to write

$$\mathbb{L}_5 = \log\mathbb{C}_5 = i\omega\big\{2(\mathbb{P}_1 - \mathbb{P}_2) + (\mathbb{P}_4 - \mathbb{P}_5)\big\}$$

(which is[3] a real antisymmeric Kirchhoffian matrix) and from it to construct the interpolating Markovia rotation matrix

$$\begin{aligned}\mathbb{C}_5(t) &= \exp\big\{t\log\mathbb{L}_5\big\}\\ &= \big\{\cos(2\omega t)\,\mathbb{I} + \sin(2\omega t)\mathbb{A}_1\big\}\mathbb{Q}_1 + \mathbb{P}_3 + \big\{\cos(\omega t)\,\mathbb{I} + \sin(\omega t)\mathbb{A}_2\big\}\mathbb{Q}_2\end{aligned}$$

where

$$\begin{aligned}\mathbb{Q}_1 &= \quad\mathbb{P}_1 + \mathbb{P}_2\\ \mathbb{A}_1 &= i\,(\mathbb{P}_1 - \mathbb{P}_2)\\ \mathbb{Q}_2 &= \quad\mathbb{P}_4 + \mathbb{P}_5\\ \mathbb{A}_2 &= i\,(\mathbb{P}_4 - \mathbb{P}_5)\end{aligned}$$

entail $\mathbb{A}_1\mathbb{Q}_1 = \mathbb{A}_1$, $\mathbb{A}_2\mathbb{Q}_2 = \mathbb{A}_2$. The matrices $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ are[3] real symmetric matrices, while $\{\mathbb{A}_1, \mathbb{A}_2\}$ are[3] real and antisymmetric. $\{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{P}_3\}$ comprise a complete set of orthogonal projection matrices. From $\mathrm{tr}\,\mathbb{P}_3 = 1$ we see that $\mathbb{P}_3$ projects onto a ray in the real vector space $\mathcal{V}_5$, namely $(1,1,1,1,1)$, while $\{\mathbb{Q}_1, \mathbb{Q}_2\}$ project onto orthogonal 2-spaces that are both orthogonal to that "axial" ray. $\mathbb{C}_5(t)$ executes a spin with angular velocity $\omega_1 = \frac{2}{5}2\pi$ on the eigenplane of $\mathbb{Q}_1$, and a spin with angular velocity $\omega_2 = \frac{1}{5}2\pi = \frac{1}{2}\omega_1$ on the eigenplane of $\mathbb{Q}_2$. $\mathbb{C}_5(t)$ gives back powers of $\mathbb{C}_5$ at integral values of t.

The emergent pattern. We have been looking to permutation matrices of the cyclic form

$$\mathbb{C}_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

---

[3] To establish such facts I used the post-command `/.`$\omega \to 2\pi/5$`//N//Chop` to convert strings of symbolic expressions into recognizable numbers.

from which we have constructed interpolating "fractional permutations"

$$\mathbb{C}_n(t) = \exp\left\{t\log\mathbb{C}_n\right\}$$

The eigenvalues of such matrices are n^{th} roots of unity $\{1, e^{i\omega}, e^{2i\omega}, \ldots, e^{(n-1)i\omega}\}$ where $\omega = 2\pi/n$. We have learned, however, to distinguish two cases: we have

$$\det\mathbb{C}_n = \prod_{\text{all roots}}(n^{\text{th}}\text{ root of unity}) = \begin{cases} -1 & : \quad n = 2\nu \qquad \text{even} \\ +1 & : \quad n = 2\nu+1 \text{ odd}\end{cases}$$

which follow as special cases (set $t = 1$) from the more general statement

$$\det\mathbb{C}_n(t) = \prod_{\text{all roots}}(n^{\text{th}}\text{ root of unity})^{\,t} = e^{i(n-1)\pi t}$$

Evidently $\det\mathbb{C}_{\text{even}}(t)$ progresses $\{+1,-1,+1,-1,\ldots\}$ as advances through the integers (*i.e.*, as $\mathbb{C}_{\text{even}}(t)$ assumes the value of successive powers of $\mathbb{C}_n$), while $\mathbb{C}_{\text{odd}}(t)$ progresses $\{+1,+1,+1,+1,\ldots\}$.

$\mathbb{C}_{2\nu}$ is unitary; it resolves the complex vector space $\mathcal{V}_{2\nu}$ into ν orthogonal planes on which it executes unitary transformations with "angular velocity parameters" $\{\omega, 2\omega, \ldots, \nu\omega\} : \omega = 2\pi/2\nu = \pi/\nu$. $\mathbb{C}_{2\nu+1}(t)$, on the other hand, is rotational; it resolves the real vector space $\mathcal{V}_{2\nu+1}$ into an invariant "axis" and ν mutually orthogonal planes each of which is orthogonal to that axis, and on which it executes rotations with angular velocities $\{\omega, 2\omega, \ldots, \nu\omega\}$: $\omega = 2\pi/(2\nu+1)$.

3×3 representation of Perm(3). The construction

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \mathbb{C}_2(t)\begin{pmatrix} x_1 \\ x_2\end{pmatrix}$$

describes (at $t = 0, 1$) the explicit action of the elements of Perm(2) on the symbols $\{x_1, x_2\}$. The construction

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t)\end{pmatrix} = \mathbb{C}_3(t)\begin{pmatrix} x_1 \\ x_2 \\ x_3\end{pmatrix}$$

provides, on the other hand, an explicit account of the action only of a cyclic *subgroup* of Perm(3), as was demonstrated on page 6. We stand in need of a parameterized 3×3 matrix that ranges over *all* the elements of Perm(3). We look in this light to the matrix produced by the "telescopic construction"

$$\mathbb{G}_3(s,t) \equiv \begin{pmatrix} \mathbb{C}_2(s) & 0 \\ 0 & 1\end{pmatrix}\mathbb{C}_3(t)$$

which is of the form

$$\mathbb{G}_3(s,t) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

but is much too complicated to spell out on the printed page, so will be allowed to remain in *Mathematica*'s memory. However, basic information follows directly from the manner in which $\mathbb{C}_2(s)$ and $\mathbb{C}_3(t)$, with their known properties, enter into the construction of $\mathbb{G}_3(s,t)$. For example, $\mathbb{G}_3(s,t)$ is a product of unitary Markov matrices, so is itself such a matrix. One has

$$\det \mathbb{G}_3(s,t) = \det \mathbb{C}_2(s) \cdot \det \mathbb{C}_3(t) = e^{i\pi s}$$

$\mathbb{G}_3(s,t)$ is evidently doubly periodic

$$\mathbb{G}_3(s+2,t) = \mathbb{G}_3(s,t+3) = \mathbb{G}_3(s,t)$$

which is to say: the $\{s,t\}$-parameter space $\mathcal{T}_2$ is toroidal.

With *Mathematica*'s assistance we look to the triples

$$\begin{pmatrix} x_1(s,t) \\ x_2(s,t) \\ x_3(s,t) \end{pmatrix} = \mathbb{G}_3(s,t) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{0,0} \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}_{0,1} \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}_{0,2} \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix}_{1,0} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix}_{1,1} \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}_{1,2}$$

where the subscripts indicate the integral values of $\{s,t\}$ that produced the triple in question. For non-integral values of the parameters $x(s,t)$ interpolates (unitarily) between those triples. Those eight triples are precisely the triples produced from $\{x_1,x_2,x_3\}$ by the action of Perm(3). Since $\mathbb{G}_3(s,t)$ is Markovian we have

$$x_1(s,t) + x_2(s,t) + x_3(s,t) = x_1 + x_2 + x_3 \quad : \quad \text{all } \{s,t\}$$

which lends an expanded meaning to the sense in which $x_1 + x_2 + x_3$ is a "symmetric polynomial," and indeed: places in an expanded context the entire elegant theory of symmetric polynomials (in three variables).

The problem addressed above—How to construct an $n \times n$ representation of Perm(n)?—arises at very n. I look to the case $n = 4$. The "telescopic procedure" employed above motivates the construction of

$$\begin{aligned} \mathbb{G}_4(s,t,u) &\equiv \begin{pmatrix} \mathbb{C}_2(s) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{C}_3(t) & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}_4(u) \\ &= \begin{pmatrix} \mathbb{G}_3(s,t) & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}_4(u) \end{aligned}$$

This is a product of 4×4 Markovian matrices with period 6 and 4, respectively. We expect it to be, therefore, Markovian with period 24 = order of Perm(4). And so, indeed, it is: on introduction of

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

the command

$$\texttt{Flatten[Table[MatrixForm[}\mathbb{G}_4(s,t,u)\texttt{.}X\texttt{]],}\{s,1,2\}\texttt{,}\{t,1,3\}\texttt{,}\{u,1,4\}\texttt{]}$$

produces a list of 24 column vectors with permuted subscripts, all of which are reported by `DeleteDuplicates` to be distinct. Moreover, the command

$$\texttt{Total[Transpose[}\mathbb{G}_4(s,t,u)\texttt{.}X\texttt{]}[\![1]\!]\texttt{]//Simplify}$$

produces[4]

$$x_1 + x_2 + x_3 + x_4 \quad : \quad \text{all } \{s,t,u\}$$

From the evident triple periodicity of $\mathbb{G}_4(s,t,u)$

$$\mathbb{G}_4(s+2,t,u) = \mathbb{G}_4(s,t+3,u) = \mathbb{G}_4(s,t,u+4) = \mathbb{G}_4(s,t,u)$$

we see that the $\{s,t,u\}$-parameter space $\mathcal{T}_3$ is hyper-toroidal. The elements of the discrete group Perm(4) live at the integral points of $\mathcal{T}_3$. Finally, we have

$$\begin{aligned} \det \mathbb{G}_4(s,t,u) &= \det \mathbb{C}_2(s)\cdot \det \mathbb{C}_3(t)\cdot \det \mathbb{C}_4(u) \\ &= e^{i\pi s}\cdot e^0 \cdot e^{i\pi u} \\ &= e^{i\pi(s+u)} \end{aligned}$$

which at integral points is ± 1 according as $s+u$ is even or odd (*i.e.*, according as s and u have the same or opposite parity) and is independent of t.

The telescopic procedure

$$\begin{aligned} \mathbb{G}_n(v_1,v_2,\ldots,v_{n-1}) \to \mathbb{G}_{n+1}&(v_1,v_2,\ldots,v_v) \\ &= \begin{pmatrix} \mathbb{G}_n(v_1,v_2,\ldots,v_{n-1}) & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}_n(v_n) \end{aligned}$$

advances dimension by 1 and (on the evidence of those examples) multiplies the period by a factor of n, resulting in a period of $n!$ = order of Perm(n). I have high confidence that $\mathbb{G}_n(v_1,v_2,\ldots,v_{n-1})$ can fairly be said to provide an $n\times n$ fractional representation of Perm(n), and at the same time an interpolated account of the explicit *action* of Perm(n) on the symbols $\{x_1,x_2\ldots,x_n\}$. We

[4] The elements of $\mathbb{G}_4(s,t,u)$ are enormously (!) complicated, but it takes *Mathematica* not more than 2 seconds to perform those calculations.

expect $\det \mathbb{G}_n(v_1, v_2, \ldots, v_{n-1})$—the continuous analog of the parity of the elements of Perm(n)—to be given by

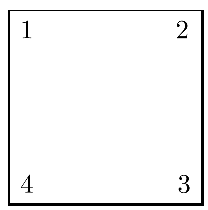
$$\det \mathbb{G}_n(v_1, v_2, \ldots, v_{n-1}) = e^{i\pi(v_1+v_3+\cdots+v_p)}$$

where $p =$ greatest odd integer $\leqslant (n-1)$.

Some unexplored ideas. From the elements $\{g_1, g_2, \ldots, g_n\}$ ($g_1 =$ identity) of a finite group construct the group table, the successive rows of which

$$k^{\text{th}} \text{ row} \quad : \quad g_k \cdot \{g_1, g_2, \ldots, g_n\}$$

are permutations of the first row, which is the upshot of Cayley's theorem: all finite groups are isomorphic to subgroups of Perm($m_{\text{sufficiently large}}$). Look, for example, to the square group (dihedral group of order 8), which describes the symmetries of the following figure:



We have

$$(1,2,3,4) \longrightarrow \begin{cases} (1,2,3,4) & \text{by identity } \mathbb{I} \\ (4,1,2,3) & \text{by } 90^\circ \text{ clockwise rotation } \mathbb{R}_1 \\ (3,4,1,2) & \text{by } 180^\circ \text{ clockwise rotation } \mathbb{R}_2 \\ (2,3,4,1) & \text{by } 270^\circ \text{ clockwise rotation } \mathbb{R}_3 \\ (4,3,2,1) & \text{by reflection in horizontal axis } \mathbb{H} \\ (2,1,4,3) & \text{by reflection in vertical axis } \mathbb{V} \\ (3,2,1,4) & \text{by reflection in ascending diagonal } \mathbb{D}_1 \\ (1,4,3,2) & \text{by reflection in descending diagonal } \mathbb{D}_2 \end{cases}$$

which are accomplished by the matrices shown on the next page. Those matrices live, as noted, at an 8-element subset of the 24 integral points $\{i, j, k\}$ on $\mathcal{T}_3$. We note that the first four of those operations (represented by $\mathbb{I}, \mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3$) reside on the curve $(0,0,u)$, while the latter four (represented by $\mathbb{H}, \mathbb{V}, \mathbb{D}_1, \mathbb{D}_2$) reside on the curve $(1,2,u)$. It is tempting to suppose that

$$\{\mathbb{G}_3(0,0,u), \mathbb{G}_3(1,2,u)\}$$

provides a continuous generalization of the square group (note in this connection that the permutations of a card can be accomplished by continuous rotations about five axes in 3-space), and that all subgroups of Perm(3) can be associated with curves drawn on $\mathcal{T}_3$.

$$\mathbb{I} \;= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{G}_4(0,0,0)$$

$$\mathbb{R}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathbb{G}_4(0,0,1)$$

$$\mathbb{R}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \mathbb{G}_4(0,0,2)$$

$$\mathbb{R}_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbb{G}_4(0,0,3)$$

$$\mathbb{H} \;= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbb{G}_4(1,2,3)$$

$$\mathbb{V} \;= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathbb{G}_4(1,2,1)$$

$$\mathbb{D}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{G}_4(1,2,0)$$

$$\mathbb{D}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \mathbb{G}_4(1,2,2)$$

Familiarly, the square group is non-Abelian, which is to say: the group table

	$\mathbb{I}$	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{R}_3$	$\mathbb{H}$	$\mathbb{V}$	$\mathbb{D}_1$	$\mathbb{D}_2$
$\mathbb{I}$	$\mathbb{I}$	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{R}_3$	$\mathbb{H}$	$\mathbb{V}$	$\mathbb{D}_1$	$\mathbb{D}_2$
$\mathbb{R}_1$	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{R}_3$	$\mathbb{I}$	$\mathbb{D}_2$	$\mathbb{D}_1$	$\mathbb{H}$	$\mathbb{V}$
$\mathbb{R}_2$	$\mathbb{R}_2$	$\mathbb{R}_3$	$\mathbb{I}$	$\mathbb{R}_1$	$\mathbb{V}$	$\mathbb{H}$	$\mathbb{D}_2$	$\mathbb{D}_1$
$\mathbb{R}_3$	$\mathbb{R}_3$	$\mathbb{I}$	$\mathbb{R}_1$	$\mathbb{R}_2$	$\mathbb{D}_1$	$\mathbb{D}_2$	$\mathbb{V}$	$\mathbb{H}$
$\mathbb{H}$	$\mathbb{H}$	$\mathbb{D}_1$	$\mathbb{V}$	$\mathbb{D}_2$	$\mathbb{I}$	$\mathbb{R}_2$	$\mathbb{R}_1$	$\mathbb{R}_3$
$\mathbb{V}$	$\mathbb{V}$	$\mathbb{D}_2$	$\mathbb{H}$	$\mathbb{D}_1$	$\mathbb{R}_2$	$\mathbb{I}$	$\mathbb{R}_3$	$\mathbb{R}_1$
$\mathbb{D}_1$	$\mathbb{D}_1$	$\mathbb{V}$	$\mathbb{D}_2$	$\mathbb{H}$	$\mathbb{R}_3$	$\mathbb{R}_1$	$\mathbb{I}$	$\mathbb{R}_2$
$\mathbb{D}_2$	$\mathbb{D}_2$	$\mathbb{H}$	$\mathbb{D}_1$	$\mathbb{V}$	$\mathbb{R}_1$	$\mathbb{R}_3$	$\mathbb{R}_2$	$\mathbb{I}$

is not symmetric (though it possesses symmetric sub-blocks, indicating the commutativity of $\{\mathbb{R}_1, \mathbb{R}_2, \mathbb{R}_3\}$, $\{\mathbb{H}, \mathbb{V}\}$ and $\{\mathbb{D}_1, \mathbb{D}_2\}$). This brings into focus the fact that while all matrices of the form $\mathbb{C}_n(t)$ (any n) commute, the matrices $\mathbb{G}_n(v_1, v_2, \ldots, v_{n-1})$—formed telescopically from them—generally do not. We confront therefore this fundamental question: **How do fractional permutations compose?** This is a difficult question to which I will return. . . but first I would like to empty my head of some additional unexplored—and quite possibly frivilous—ideas.

In the quantum theory of indistinguishable particles one encounters objects of the form

$$\begin{aligned}\Psi_S(x_1, x_2, \ldots, x_n) &= \sum_{\mathcal{P}} \psi(\mathcal{P}[x_1, x_2, \ldots, x_n])\\ \Psi_A(x_1, x_2, \ldots, x_n) &= \sum_{\mathcal{P}} (-)^{\mathcal{P}} \psi(\mathcal{P}[x_1, x_2, \ldots, x_n])\end{aligned}$$

—"symmetrized/antisymmetrized" versions of the function $\psi(x_1, x_2, \ldots, x_n)$. Often one assumes $\psi(x_1, x_2, \ldots, x_n)$ to have the form

$$\psi(x_1, x_2, \ldots, x_n) = \phi_{i_1}(x_1)\phi_{i_2}(x_2)\cdots\phi_{i_n}(x_n)$$

where the $\phi_i(x)$ are orthonormal elements of a Hilbert space $\mathcal{H}$ and—writing $\mathcal{F}_n$ to denote the space of all such (anti)symmetrized product functions—constructs the Fock space $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots$ and introduces operators a_i and a_i^+ that (respectively) remove $\phi_i(x_n)$-dependence from elements of $\mathcal{F}_k$ to produce elements of $\mathcal{F}_{k-1}$ or introduce $\phi_i(x_n)$-dependence to produce elements of $\mathcal{F}_{k+1}$.[5] I am led semi-whimsically to wonder: Is it feasible to

$$\begin{aligned}&\text{replace } \sum_{\mathcal{P}} &&\text{by } \int_{\mathcal{G}} &&?\\ &\text{replace } \sum_{\mathcal{P}}(-)^{\mathcal{P}} &&\text{by } \int_{\mathcal{G}} \det(\mathcal{G}) &&?\end{aligned}$$

Having wandered into the quantum regime, I note that modern quantum theory has given rise the development of ideas alternative to those we associate with the names of Maxwell-Boltzmann, Einstein-Bose and Fermi-Dirac, several of which are known as "fractional statistics." I am thinking of "parastatistics"

[5] In "Toy quantum field theory: populations of indistinguishable finite-state systems," (notes for a Reed Physics Seminar presented on 1 November 2000), I show in detail how such a formalism works out when the $\phi_i(x)$ are replaced by orthonormal basis vectors in a finite-dimensional vector space and products are replaced by Kronecker products. I apply the resulting formalism illustratively to the quantum theory of angular momentum, and obtain results that somewhat resemble results obtained by Majorana and Schwinger.

(which engaged the productive interest of O. W. Greenberg, my friend from Brandeis days), "braid statistics," "anyon statistics (Abelian & non-Abelian)" which inter into the theory of the fractional quantum Hall effect and engage notions like fractional spin, fractional charge, topological field theory, Berry phase and similar exotica. One cannot but wonder whether "fractional permutations" have a role to play (perhaps already do play?) in such subject areas.

The composition problem for fractional permutations. The permutation group Perm(n) is—as a group—certainly closed under composition. The set of all $n \times n$ Markov matrices is also closed under composition.[6] $\mathbb{G}_n(\xi)$-matrices[7] are Markovian and have been seen—at least in the cases $\mathbb{G}_3$(integers) and $\mathbb{G}_4$(integers), where they reproduce the action of Perm(3) and Perm(4)—to be multiplicatively closed. It is natural, therefore, to anticipate the existence of formulae of the type

$$\mathbb{G}_n(\xi_1) \cdot \mathbb{G}_n(\xi_2) = \mathbb{G}_n(\xi_n(\xi_1, \xi_2))$$

where $\xi(\xi_1, \xi_2)$ provides the fractional extension of the Perm(n) group table,[8] to which it reduces when ξ_1 and ξ_2 are both integral.

Composition problems of this sort are usually difficult. For example: the composition of 3×3 rotation matrices entails finding $\{\theta, \mathbb{A}\}$ such that

[6] But not a group: it contains an identity, but the inverses of Markov matrices are typically—but not invariably—non-Markovian (columns sum to unity, but some elements may fall outside the interval $[0, 1]$).

[7] Here I have adopted the abbreviation $\xi \equiv \{v_1, v_2, \ldots, v_{n-1}\}$.

[8] *Mathematica*'s `MinimumChangePermutations[{1,2,3}]`, which requires installation of the the `Combinatorica` package, produces

$$\begin{array}{lll} p_1 = \{1,2,3\} & p_3 = \{3,1,2\} & p_5 = \{2,3,1\} \\ p_2 = \{1,2,3\} & p_4 = \{1,3,2\} & p_6 = \{3,2,1\} \end{array}$$

Writing (for example) $\{2,3,1\}\{a,b,c\} = \{b,c,a\}$ to signify that the first factor has acted upon the second (accomplished by `Permute[{`a,b,c`},{`$2,3,1$`}]`), we obtain for Perm(3) this illustrative group table:

	p_1	p_2	p_3	p_4	p_5	p_6
p_1	p_1	p_2	p_3	p_4	p_5	p_6
p_2	p_2	p_1	p_4	p_3	p_6	p_5
p_3	p_3	p_6	p_5	p_2	p_1	p_4
p_4	p_4	p_5	p_6	p_1	p_2	p_3
p_5	p_5	p_4	p_1	p_6	p_3	p_2
p_6	p_6	p_3	p_2	p_5	p_4	p_1

That is the kind of information we expect $\xi_3(\xi_1, \xi_2)$ to provide. And, moreover, to interpolate!

$$e^{\theta_1 \mathbb{A}_1} \cdot e^{\theta_2 \mathbb{A}_2} = e^{\theta \mathbb{A}}$$

where in the present instance the $\mathbb{A}$-matrices are 3×3 antisymmetric. The problem becomes difficult when $\mathbb{A}_1$ and $\mathbb{A}_2$ fail to commute (or at least to commute with their commutator). The generalized spectral resolution provides in principle a method of attack, but generally leads to results in which it is difficult to discover a coherent pattern.[9]

I restrict my exploratory remarks to the simplest case: the composition problem posed by $\mathbb{G}_3(s,t)$.[10]

The eigenvalues of $\mathbb{G}_3(s,t)$, as supplied by *Mathematica*, are complicated, but can be brought finally to the form

$$\left.\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= \alpha - \sqrt{\alpha^2 - e^{i\pi s}} \\ \lambda_3 &= \alpha + \sqrt{\alpha^2 - e^{i\pi s}} \end{aligned}\right\} \quad : \quad \alpha = e^{i\frac{1}{2}\pi s} \cos\tfrac{1}{2}\pi s\, \cos\tfrac{2}{3}\pi t \tag{7}$$

which are, we happen to notice, the roots of

$$(x-1)(x^2 - 2\alpha x + e^{i\pi s}) = 0$$

Interestingly, they would yield the correct product

$$\det \mathbb{G}_3(s,t) = \lambda_1\lambda_2\lambda_3 = e^{i\pi s}$$

whatever real or complex value were assigned to α. Elementary manipulations give

$$\begin{aligned} \lambda_2 &= e^{i\frac{1}{2}\pi s}\Big\{(\cos\tfrac{1}{2}\pi s\, \cos\tfrac{2}{3}\pi t) - i\sqrt{1 - (\cos\tfrac{1}{2}\pi s\, \cos\tfrac{2}{3}\pi t)^2}\,\Big\} \\ \lambda_3 &= e^{i\frac{1}{2}\pi s}\Big\{(\cos\tfrac{1}{2}\pi s\, \cos\tfrac{2}{3}\pi t) + i\sqrt{1 - (\cos\tfrac{1}{2}\pi s\, \cos\tfrac{2}{3}\pi t)^2}\,\Big\} \end{aligned}$$

whence

$$\begin{aligned} \lambda_2 &= e^{i[\frac{1}{2}\pi s - \varphi(s,t)]} \\ \lambda_3 &= e^{i[\frac{1}{2}\pi s + \varphi(s,t)]} \end{aligned} \tag{8.1}$$

where

$$\begin{aligned} \varphi(s,t) &= \arccos\,(\cos\tfrac{1}{2}\pi s\, \cos\tfrac{2}{3}\pi t) \\ &= \arccos\Big\{\tfrac{1}{2}\big[\cos\pi\big(\tfrac{1}{2}s + \tfrac{2}{3}t\big) + \cos\pi\big(\tfrac{1}{2}s - \tfrac{2}{3}t\big)\big]\Big\} \end{aligned} \tag{8.2}$$

Equations (7) conform to the fact that the eigenvalues of unitary matrices lie always on the unit circle. At $t = 0$ they supply $\lambda_2 = e^{i\pi s}$ and $\lambda_3 = 1$.

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[9] This problem, as it relates to 4×4 Lorentz matrices, is—after many pages of dense algebraic preparation—addressed and solved on pages 75–94 of *Elements of Special Relativity* (1966).

[10] The case $\mathbb{G}_2(s) \equiv \mathbb{C}_2(s)$ is trivial, since it is cyclic.

The eigenvalues of $\mathbb{G}_3(s,t;\sigma,\tau) \equiv \mathbb{G}_3(s,t)\cdot\mathbb{G}_3(\sigma,\tau)$, as supplied to us by *Mathematica*, are $\{1,\lambda_2(s,t;\sigma,\tau),\lambda_3(s,t;\sigma,\tau)\}$ where

$$\lambda_2 = (\text{sum of 9 terms}) - \sqrt{(\text{sum of 48 terms})}$$

from which λ_3 is obtained by a simple sign reversal, and where the terms are real or imaginary 4-fold products of sines or cosines (or their squares) of $\{\pi s, \pi\sigma, \frac{2}{3}\pi t, \frac{2}{3}\pi\tau\}$; they are, in short, a complicated mess. But they do conform to the requirement that

$$\det\mathbb{G}_3(s,t)\mathbb{G}_3(\sigma,\tau) = \lambda_2\lambda_3 = \det\mathbb{G}_3(s,t)\cdot\det\mathbb{G}_3(s,t) = e^{i\pi(s+\sigma)}$$

and yield to simplifications patterned on (7). We find at length that

$$\begin{aligned}
\lambda_1(s,t,\sigma,\tau) &= 1\\
\lambda_2(s,t,\sigma,\tau) &= a - \sqrt{a^2 - e^{i\pi(s+\sigma)}}\\
\lambda_3(s,t,\sigma,\tau) &= a + \sqrt{a^2 - e^{i\pi(s+\sigma)}}
\end{aligned}$$

where

$$a = e^{i\frac{1}{2}\pi(s+\sigma)}A$$

and

$$\begin{aligned}
A(s,t,\sigma,\tau) = \tfrac{1}{4}\big\{\quad &\cos\pi\big[\tfrac{1}{2}(s+\sigma)+\tfrac{2}{3}(t+\tau)\big] + \cos\pi\big[\tfrac{1}{2}(s+\sigma)+\tfrac{2}{3}(t-\tau)\big]\\
+&\cos\pi\big[\tfrac{1}{2}(s+\sigma)-\tfrac{2}{3}(t+\tau)\big] + \cos\pi\big[\tfrac{1}{2}(s+\sigma)-\tfrac{2}{3}(t-\tau)\big]\\
+&\cos\pi\big[\tfrac{1}{2}(s-\sigma)+\tfrac{2}{3}(t+\tau)\big] - \cos\pi\big[\tfrac{1}{2}(s-\sigma)+\tfrac{2}{3}(t-\tau)\big]\\
+&\cos\pi\big[\tfrac{1}{2}(s-\sigma)+\tfrac{2}{3}(t+\tau)\big] - \cos\pi\big[\tfrac{1}{2}(s-\sigma)-\tfrac{2}{3}(t-\tau)\big]\big\}\\
= \tfrac{1}{2}\big\{\quad &\cos\pi\big[\tfrac{1}{2}(s+\sigma)\big]\cos\pi\big[\tfrac{2}{3}(t+\tau)\big]\\
+&\cos\pi\big[\tfrac{1}{2}(s+\sigma)\big]\cos\pi\big[\tfrac{2}{3}(t-\tau)\big]\\
+&\cos\pi\big[\tfrac{1}{2}(s-\sigma)\big]\cos\pi\big[\tfrac{2}{3}(t+\tau)\big]\\
-&\cos\pi\big[\tfrac{1}{2}(s-\sigma)\big]\cos\pi\big[\tfrac{2}{3}(t-\tau)\big]\big\}
\end{aligned}$$

give

$$\begin{aligned}
\left.\begin{matrix}\lambda_2\\ \lambda_3\end{matrix}\right\} &= e^{i\frac{1}{2}\pi(s+\sigma)}\big\{A \mp i\sqrt{1-A^2}\big\}\\
&= e^{i[\frac{1}{2}\pi(s+\sigma)\mp\Phi(s,t,\sigma,\tau)]} \qquad (9.1)\\
\Phi(s,t,\sigma,\tau) &= \arccos[A(s,t,\sigma,\tau)] \qquad (9.2)
\end{aligned}$$

From (9) we recover (8) at $\sigma=\tau=0$, as we should, since $\mathbb{G}_3(0,0)=\mathbb{I}$. The periodicity properties of $\Phi(s,t,\sigma,\tau)$ duplicate (in the sense "provide two copies of") those of $\varphi(s,t)$, just as the periodicity properties of $\mathbb{G}_3(s,t;\sigma,\tau)$ duplicate those of $\mathbb{G}_3(s,t)$.

The results (9), though not simple, are at any rate attractively patterned, which encouraged me to think that the description of $\mathbb{G}_3(s,t;\sigma,\tau)$ might also be, and might—in the sense that (9.1) mimics the structure of (8.1)—mimic the structure of $\mathbb{G}_3(s,t)$, which at page 13 had been allowed to remain within *Mathematica*'s memory, but of which I undertook to make patterned sense. When numerical values are assigned arbitrarily to $\{s,t\}$ *Mathematica* reports that the eigenvectors corresponding to the eigenvalues $\{1,\lambda_1,\lambda_2\}$ to be of the forms

$$\begin{pmatrix}1\\1\\1\end{pmatrix},\quad \begin{pmatrix}\alpha_2 \qquad + i\beta_2\\ -(\alpha_2+1)-i\beta_2\\ 1\end{pmatrix},\quad \begin{pmatrix}\alpha_3 \qquad + i\beta_3\\ -(\alpha_3+1)-i\beta_3\\ 1\end{pmatrix}$$

respectively, where the elements of the latter two sum to zero, and which altogether comprise an unnormalized orthogonal triad. The problem, therefore, is to make structured symbolic sense of $a_{21}=\alpha_2+i\beta_2$. But a_{21} is enormously complicated, and an intense effort to make sense of it came to naught; my effort to construct a patterned symbolic description of $\mathbb{G}_3(s,t)$ was frustrated. The problem here traces to the circumstance that $\mathbb{G}_3(s,t)$ is by construction a product, and the spectral properties of products stand in an obscure relationship to those of the factors (except when they commute). We can expect that problem to be exacerbated when we turn from $\mathbb{G}_3(s,t)$ to $\mathbb{G}_3(s,t)\cdot\mathbb{G}_3(\sigma,\tau)$. The problem presented by $\mathbb{G}_4(s,t,u)\cdot\mathbb{G}_4(\sigma,\tau,\upsilon)$ is even worse, while that presented by the composition law for the fractional extension of Perm(n) ($n\geqslant 5$) is quite out of reach, since *Mathematica* does not provide symbolic solutions of quintics or higher order polynomials.

It is, however, perhaps worth mentioning that if one assigns (integral or non-integral) values to s (else t) things simplify greatly. For example, we have

$$\mathbb{G}_3(1,t)=\tfrac{1}{3}\begin{pmatrix}1-C+\sqrt{3}S & 1+2C & 1-C-\sqrt{3}S\\ 1+2C & 1-C-\sqrt{3}S & 1-C+\sqrt{3}S\\ 1-C-\sqrt{3}S & 1-C+\sqrt{3}S & 1+2C\end{pmatrix}$$

$$C=\cos\tfrac{2}{3}\pi t,\quad S=\sin\tfrac{2}{3}\pi t$$

which by $\det\mathbb{G}_3(1,t)=-1$ (all t) is seen to interpolate between the odd elements of Perm(3); *i.e.*. to trace a curve on $\mathfrak{T}_2$. Similarly,

$$\mathbb{G}_3(0,t)=\tfrac{1}{3}\begin{pmatrix}1+2C & 1-C-\sqrt{3}S & 1-C+\sqrt{3}S\\ 1-C+\sqrt{3}S & 1+2C & 1-C-\sqrt{3}S\\ 1-C-\sqrt{3}S & 1-C+\sqrt{3}S & 1+2C\end{pmatrix}$$

is seen by $\det\mathbb{G}_3(1,t)=+1$ to interpolate between the even elements of Perm(3). Those matrices are simple enough to admit of feasible symbolic spectral decomposition. Explicit composition is made simple in the latter case by the fact that $\mathbb{G}_3(0,t)$ and $\mathbb{G}_3(0,\tau)$ commute, but difficult in the former case because $\mathbb{G}_3(1,t)$ and $\mathbb{G}_3(1,\tau)$ generally fail to commute. Similar remarks pertain to $\mathbb{G}_3(s,x)$ and $\mathbb{G}_3(\sigma,x)$ unless $x\equiv 0 \bmod 3$.

ADDENDUM

Comparison with an alternative procedure. On pages 4 and 5 I labored (brought into play the full "generalized spectral decomposition" apparatus) to construct the fractional generalization (8) of the simple permutation matrix

$$\mathbb{C}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It has occurred to me—while working on quite another problem—that the Markovian matrix $\mathbb{C}_2$ can be written

$$\begin{aligned} \mathbb{C}_2 &= \mathbb{I} + \mathbb{K} \\ \mathbb{K} &= \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \text{ is a Kirchhoff matrix} \end{aligned}$$

It seems natural, therefore, to proceed directly to the construction of

$$e^{t\,\mathbb{K}} = \tfrac{1}{2}\begin{pmatrix} 1+e^{-2t} & 1-e^{-2t} \\ 1-e^{-2t} & 1+e^{-2t} \end{pmatrix}$$

which—though manifestly Markovian—is *not* cyclic. A similar result is obtained if (for example) one writes

$$\begin{aligned} \mathbb{C}_3 &= \mathbb{I} + \mathbb{K} \\ \mathbb{K} &= \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \end{aligned}$$